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MAT 219 (Final Exam Review)

Chapter 6 (Laplace Transforms)
- → Formulas to memorize
- → Skills

Chapter 10 (Heat Eqn. & Fourier Series)
- → Formulas to memorize
- → Skills

Laplace Transforms of Functions (Memorize!)

Definition: $\mathcal{L}\{f(t)\} = \int_0^\infty f(t)\cdot e^{-st}\,dt$

Property: $\mathcal{L}\{\frac{d}{dt} f(t)\} = s\,\mathcal{L}\{f(t)\} - f(0)$

Basic Formulas:

$\mathcal{L}\{1\} = 1/s$

$\mathcal{L}\{t\} = 1/s^2$

$\left.\right\}\ \mathcal{L}\{t^n\} = n!/s^{n+1}$

$\mathcal{L}\{\cos bt\} = \frac{s}{s^2+b^2}$

$\mathcal{L}\{\sin bt\} = \frac{b}{s^2+b^2}$

$\boxed{\begin{array}{l} \ell^a = \text{"lag operator"} \\ \ell^a[f(t)] = f(t-a) \\ \ell^a[F(s)] = F(s-a) \end{array}}$

Exponential Shift:

$\mathcal{L}\{e^{at} f(t)\} = \ell^a[\mathcal{L}\{f(t)\}]$

Inverse: $\mathcal{L}^{-1}\{F(s)\} = e^{at}\mathcal{L}^{-1}\{F(s+a)\}$

Step Functions:

$\mathcal{L}\{u_c(t)\cdot f(t)\} = e^{-cs}\mathcal{L}\{f(t+c)\}$

Inverse: $\mathcal{L}^{-1}\{e^{-cs}F(s)\} = u_c(t)\cdot \ell^c[\mathcal{L}^{-1}\{F(s)\}]$

Impulse Functions:

$\mathcal{L}\{\delta(t-c)\} = e^{-cs}$

$\mathcal{L}\{\delta(t-c)\cdot f(t)\} = e^{-cs}\underbrace{f(c)}_{\text{constant!}}$

Note:

$\mathcal{L}^{-1}\{e^{-cs}\} = \delta(t-c)$

$\mathcal{L}^{-1}\{\frac{e^{-cs}}{s}\} = u_c(t)$

$\mathcal{L}^{-1}\{\frac{e^{-cs}}{s^2}\} = u_c(t)\cdot(t-c)$

→ In general, $\delta(t-c)\, f(t) = \delta(t-c)\, f(c)$

Laplace Transforms of Differential Eqns

$\mathcal{L}\{y\} = Y$

$\mathcal{L}\{y'\} = sY - y(0)$

$\mathcal{L}\{y''\} = s^2Y - sy(0) - y'(0)$

$\mathcal{L}\{y'''\} = s^3Y - s^2y(0) - sy'(0) - y''(0)$

etc.

Convolutions

Definition: $g * f = \int_0^t \underbrace{g(t-\tau)}_{\text{value at time } t \text{ of impulse resp. starting at } \tau}\ \underbrace{f(\tau)}_{\text{height of forcing function at start of impulse}}\, d\tau$

Property: $\mathcal{L}\{g*f\} = \mathcal{L}\{g\}\cdot\mathcal{L}\{f\}$

"Convolution Theorem"

$\Longrightarrow g*f = \mathcal{L}^{-1}\{\mathcal{L}\{g\}\cdot\mathcal{L}\{f\}\}$

$\boxed{\text{Note: } (f*g)(0) = 0 \text{ ALWAYS}}$

Chapter 6 Skills

- Compute $\mathcal{L}\{f(t)\}$ for a basic function using
  - Integral definition
    $$\mathcal{L}\{f(t)\} = \int_0^\infty f(t) \cdot e^{-st}\, dt$$
    $$= \lim_{R\to\infty} \int_0^R f(t) \cdot e^{-st}\, dt = \cdots$$
  - Derivative property of Laplace transforms
    $$\mathcal{L}\left\{\tfrac{d}{dt} f(t)\right\} = s\mathcal{L}\{f(t)\} - f(0)$$

- Find $\mathcal{L}$ and $\mathcal{L}^{-1}$ of functions using basic memorized formulas and rules (exponential-shift, step functions, etc)

- Rewrite piecewise-defined function in terms of step functions
  $$f = \begin{cases} f_1 & t < c_1 \\ f_2 & c_1 \le t < c_2 \\ \vdots & \end{cases} \rightsquigarrow f = f_1 + (f_2 - f_1)\, u_{c_1} + \cdots$$

- Solve a differential equation by applying $\mathcal{L}$, solving for Y, and then computing $\mathcal{L}^{-1}$!

  Don't forget initial values!!

- Use convolutions to write the solution to a differential equation

- Compute the convolution $f * g$ for basic functions using
  - Integral definition
    $$(f*g)(t) = \int_0^t f(t-\tau) \cdot g(\tau)\, d\tau$$

    (Note: This can also be computed as $\int_0^t f(\tau)\, g(t-\tau)\, d\tau$ because $f*g = g*f$.)
  - The basic property of convolutions (the "Convolution Theorem")
    $$\mathcal{L}\{f*g\} = \mathcal{L}\{f\} \cdot \mathcal{L}\{g\}$$
    - This means that
      $$f*g = \mathcal{L}^{-1}\{\mathcal{L}\{f\} \cdot \mathcal{L}\{g\}\}$$

Fourier Series Formulas (Memorize!)

Fourier Series for $f(x)$ on $-L < x < L$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right)\right)$$

where

$$\begin{cases} a_n = \dfrac{1}{L}\displaystyle\int_{-L}^{L} f(x)\cos\left(\frac{n\pi}{L}x\right)dx \\ b_n = \dfrac{1}{L}\displaystyle\int_{-L}^{L} f(x)\sin\left(\frac{n\pi}{L}x\right)dx \end{cases} \qquad \left(a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx\right)$$

(Note: cos / sin $\left(\frac{n\pi}{L}x\right)$ has frequency n over the period $-L < x < L$.)

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f even $\Longrightarrow$ **Cosine Series for $f(x)$ on $\boxed{0} < x < L$**

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right)$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi}{L}x\right)dx \qquad \left(a_0 = \frac{2}{L}\int_0^L f(x)\,dx\right)$$

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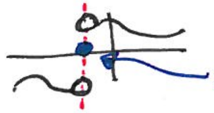
f odd $\Longrightarrow$ **Sine Series for $f(x)$ on $\boxed{0} < x < L$**

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right)$$

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx$$

Notes:

- Fourier series are periodic
  $\Longrightarrow$ value at x = value at $x \pm 2L$
- Series hit middle value at jump discontinuities



series will equal this value at jump-point.

Heat Equation

Rod of length L

$u(x,t)$ = temperature at position x $(0 \le x \le L)$
and time t $(0 \le t\ )$

At one position	At one time
u_t =	$\alpha^2 u_{xx}$
"change in temp at position x over time"	"difference between temp at x and nearby (at time t)"

Solution: If temp. at ends $= 0$ $\left(\text{i.e.} \begin{cases} u(0,t) = 0 \\ u(L,t) = 0 \end{cases}\right)$

and $f(x) = b_1 \sin\left(\frac{\pi}{L}x\right) + b_2 \sin\left(\frac{2\pi}{L}x\right) + \cdots$

then

$$\boxed{u(x,t) = b_1 \sin\left(\frac{\pi}{L}x\right)e^{-\left(\frac{\pi\alpha}{L}\right)^2 t} + b_2 \sin\left(\frac{2\pi}{L}x\right)e^{-\left(\frac{2\pi\alpha}{L}\right)^2 t} + \cdots}$$

More generally

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right) \quad \leftarrow \text{Fourier Sine Series}$$

$$\Longrightarrow \boxed{u(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right) e^{-\left(\frac{n\pi\alpha}{L}\right)^2 t}}$$

Chapter 10 Skills

- **Separation of Variables** for partial differential eqns
  1. Substitute $u(x,t) = X(x) \cdot T(t)$

     $u = X \cdot T$ $\quad u_x = X' T$ $\quad u_{xx} = X'' \cdot T$

     $u_t = X \cdot T'$ $\quad u_{tt} = X \cdot T''$ $\quad$ etc...

     $u_{xt} = X' \cdot T'$
  2. Move all (X & x) and (T & t) to opposite sides of equation

     $$\begin{pmatrix} X, X' \text{ etc} \\ \text{and } x \end{pmatrix} = \begin{pmatrix} T, T' \text{ etc} \\ \text{and } t \end{pmatrix}$$
  3. Set each side $= \lambda$. (Simplify)

- **Boundary Value Problems**
  1. Find general solution
  2. Plug in boundary values to get system of eqns
  3. Try to solve for $c_1, c_2 \ldots$
     - → Maybe no solution (If one variable must equal two <u>different</u> things)
     - → Maybe ∞ many solutions (If one variable is determined but other is free.)

- **Eigenvalue / Eigenfunction problems**
  - → Boundary value problems with λ (Find λ so that there are <u>nonzero</u> solutions)

- Compute Fourier series coefficients (and Sine & Cosine series) for simple functions using integral formulas

  $$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi}{L}x\right)dx$$

  etc.

- Write the Fourier series for a function if you know the coefficients (a_n, b_n)

- Use periodicity of Fourier series to give values at points / graph.

- Write solutions to heat equation
  - → when $u(x,0) =$ (sum of sine functions)
  - → when $u(x,0) =$ (something else)
    - → Convert to Fourier Sine series